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LETTER TO THE EDITOR

On the quantum symmetries associated with the two parameter free fermion model

M Bednář†, C Burdík‡, M Couture§ and L Hlavatý†

† Institute of Physics, Czechoslovak Academy of Sciences, Na Slavance 2, 180 40 Prague 8, Czechoslovakia
‡ Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czechoslovakia
§ Chalk River Laboratories, AECL, Chalk River, Ontario, Canada K0J 1J0

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Abstract. The quantum algebras associated with the R-matrix obtained from the two parameter free fermion model are examined. It is shown that in addition to $\mathrm{U}_{q,s}(\mathrm{gl}(1,1))$ there exists another algebra $\bar{\mathrm{U}}_{q,s}(\mathrm{gl}(1,1))$ which is isomorphic to $\mathrm{U}_{q,s}(\mathrm{gl}(1,1))$ as an algebra but has a different Hopf structure. $\bar{\mathrm{U}}_{q,s}(\mathrm{gl}(1,1))$ does not have a classical limit and is related to $\mathrm{U}_p(\mathrm{sl}(2,\mathbb{C}))$ at a root of unity ($p=\mathrm{i}, \mathrm{i}^2=-1$).

It is known that representations of the braid group can be obtained from the Boltzmann weights of solvable models. From the two parameter (we are referring to parameters other than the spectral one) free fermion model, one obtains the following R-matrix

$$\check{R}(q,s)=\begin{bmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & s^{-1} & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{bmatrix} \tag{1}$$

which is a solution of

$$\check{R}_{12}\check{R}_{23}\check{R}_{12}=\check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{2}$$

where the subscripts indicate the action on the triple tensor product $V\otimes V\otimes V$ where V is a two-dimensional complex vector space. The two parameter model is obtained from the one parameter case [1] (see for instance the one parameter model $(1,-1)$ in Deguchi's paper) by a simple symmetry breaking transformation [2]. $\check{R}(q,s)$ was first reported in [3] where it had been obtained by solving (2) directly.

It is well known [4–6] that $\check{R}(q,s=1)$ is related to the one parameter deformation of GL(1, 1). Less known is the fact that there exists another symmetry [7–10] associated with $\check{R}(q,s=1)$ that resembles that of GL(1, 1) but for which there is no classical limit. The purpose of this letter is to examine the two parameter case. Although our main focus is the Hopf algebras defined in the space dual to that of the quantum groups, it will prove useful to begin the analysis at the level of the quantum spaces where the symmetries involved are more transparent. We begin by recalling some of the features of the one parameter case as discussed in [8].

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In Manin's approach, quantum groups coact on a set of quantum spaces (quadratic algebras). In his treatment of the general linear supergroup [11] he considers a pair A and A^* of quantum spaces where A^* is dual to A. The algebraic structure he obtains corresponds to particular definitions of duality and rule for the multiplication in the tensor product of two algebras. The implications, in the one parameter case ($q_{mn}=q$ for all m, n), of a different set of definitions were examined in [8]. The space dual to that of the quantum group was defined for each set of definitions. In the two-dimensional case two distinct algebras were obtained: $\mathrm{U}_q(\mathrm{gl}(1,1))$ and the other which we denote $\bar{\mathrm{U}}_q(\mathrm{gl}(1,1))$. The latter was shown to be isomorphic to $\mathrm{U}_q(\mathrm{gl}(1,1))$ as an algebra while having a different Hopf structure. $\bar{\mathrm{U}}_q(\mathrm{gl}(1,1))$ does not have a classical limit and is related to $\bar{\mathrm{U}}_p(\mathrm{sl}(2,\mathbb{C}))$ at a root of unity ($p=\mathrm{i}, \mathrm{i}^2=-1$).

The two parameter case considered here is of the non-standard type [12]. The novelty in such cases is that one must consider the coaction on a set of two quantum spaces that are generally distinct from each other (one is not the dual of the other, see also [13]). Let us introduce the quantum spaces A and B defined as follows:

$$A=k\langle x_1, x_2\rangle/(x_2^2, x_1x_2-s^{-1}qx_2x_1) \tag{3}$$

$$B=k\langle x_1, x_2\rangle/(x_2^2, x_1x_2-qsx_2x_1) \tag{4}$$

where $k\langle x_1, x_2\rangle$ means an associative k-algebra freely generated by x_1 and x_2; k is the field. Note that A, B are generated by $(\check{R}-qI)(x\otimes x)=0$ and $(\check{R}^T-qI)(x\otimes x)=0$ respectively. In what follows we shall consider the coaction of a bialgebra on a pair of quantum spaces which includes A and the dual of B. (Note that in the case $s=1$ we have $A=B$.) Our main objective is the study of the implications (with main focus on enveloping algebras) of two distinct definitions of duality accompanied by particular choices for the multiplication rule in the tensor product of two algebras.

Let us first consider the case where duality is defined through the following pairings

$$\langle x^n; x_m\rangle=\delta^n_m \qquad \langle x^k\otimes x^l; x_m\otimes x_n\rangle=\delta^k_m\delta^l_n \tag{5}$$

Denoting by $B!$ the dual of B it follows from (5) that

$$B!=k\langle x^1, x^2\rangle/((x^1)^2, x^1x^2+q^{-1}s^{-1}x^2x^1) \tag{6}$$

We now define a bialgebra with geneators t^n_m ($m, n=1,2$) which coacts on the pair $(A, B!)$ by requiring that the following maps

$$\delta(x_m)=\sum_{j=1}^{2} t^j_m\otimes x_j \qquad \delta(x^k)=\sum_{l=1}^{2} t^l_k\otimes x^l \tag{7}$$

be homomorphisms of A and $B!$ respectively. The multiplication rule in the tensor product of two algebras is defined to be

$$(a\otimes b)(c\otimes d)=(ac\otimes bd). \tag{8}$$

Let

$$T=\begin{pmatrix} t^1_1 & t^2_1 \\ t^1_2 & t^2_2 \end{pmatrix}.$$

The quadratic relations satisfied by the t^n_m can be summarized as follows:

$$RT_1PT_1P=PT_1PT_1R \tag{9}$$

where P is the permutation operator, $R=P\check{R}$ and $T_1=T\otimes I$. The explicit form of relations that follow from (9) can be found in [13]. We denote by $\bar{\mathrm{A}}_R$ the associative algebra over $\mathbb{C}$ with generators $\mathbf{1}$ and t^n_m $(m, n=1,2)$ which satisfy (9). The coproduct is

$$\Delta(t^n_m)=\sum_{x=1}^{2} t^x_m \otimes t^n_x \qquad \Delta(1)=1\otimes 1. \tag{10}$$

A representation ρ of $\bar{\mathrm{A}}_R$ is $\rho(t^n_m)^l_k = R^{nl}_{mk}$. The coproduct is an algebra homomorphism provided one uses the rule (8) which is of course not surprising; however, later it will be shown that there exists another bialgebra one can associate to $\check{R}(q,s)$ for which the multiplication rule differs from (8). Following Fadeev *et al* [14] we now define $\bar{\mathrm{U}}_{q,s}(\mathrm{gl}(1,1))$ as a subalgebra of the dual to $\bar{\mathrm{A}}_R$; $\bar{\mathrm{U}}_{q,s}$ is generated by the unit element $1'$ and the generators $L^{(\pm)}_{mn}(m,n=1,2)$ which are defined by the following duality relations:

$$\langle \mathbf{1};\, T_1T_2\ldots T_k\rangle = 1^{\otimes k} \qquad \langle L^{(\pm)};\, T_1T_2\ldots T_k\rangle = R^{(\pm)}_1 R^{(\pm)}_2 \ldots R^{(\pm)}_k \tag{11}$$

where

$$L^{(\pm)}=\begin{pmatrix} L^{(\pm)}_{11} & L^{(\pm)}_{12} \\ L^{(\pm)}_{21} & L^{(\pm)}_{22}\end{pmatrix}$$

and I is the unit matrix. $T_m = I\otimes\ldots I\otimes T\otimes I\otimes\ldots\otimes I$ (k tensor products and T in the m position) and $R^{(\pm)}_m$ act non-trivially on factor number 0 and m in the tensor product $V^{\otimes(k+1)}$ and coincide there with the matrix $R^{(\pm)}$ defined as follows:

$$R^{(+)}=PRP \qquad R^{(-)}=R^{-1} \tag{12}$$

It follows from (11) that $L^{(+)}_{21}=L^{(-)}_{12}=0$ and the generators satisfy

$$\begin{aligned} &RP(L^{(\pm)}\otimes I)P(L^{(\pm)}\otimes I)=(L^{(\pm)}\otimes I)P(L^{(\pm)}\otimes I)PR \\ &RP(L^{(+)}\otimes I)P(L^{(-)}\otimes I)=(L^{(-)}\otimes I)P(L^{(+)}\otimes I)PR \end{aligned} \tag{13}$$

with the coproduct and antipode S defined as follows:

$$\Delta(L^{(\pm)}_{mn})=\sum_x L^{(\pm)}_{mx}\otimes L^{(\pm)}_{xn} \qquad S(L^{(\pm)})L^{(\pm)}=I. \tag{14}$$

A representation ρ is

$$\rho(L^{(+)}_{mn})^l_k = R^{ln}_{km} \qquad \rho(L^{(-)}_{mn})^l_k=(R^{-1})^{nl}_{mk}. \tag{15}$$

We introduce the generators $H, X^{\pm}$ and Z which satisfy the following relations: $[H, X^{\pm}]=\pm 2X^{\pm}$. $[H,Z]=[Z,X^{\pm}]=0$. We first connect them with $\mathrm{U}_{p=1}(\mathrm{sl}(2,\mathbb{C}))$ by making the following identification $(\mathrm{i}^2=-1)$:

$$\begin{aligned} &L^{(+)}_{11}=q^{-(H+Z)/2}s^{-(H-Z)/2} \qquad L^{(+)}_{22}=q^{-(H-Z)/2}s^{-(H+Z)/2}\,\mathrm{i}^{(H-Z)} \\ &L^{(-)}_{11}=q^{(H+Z)/2}s^{-(H-Z)/2} \qquad L^{(-)}_{22}={}^{(H-Z)/2}s^{-(H+Z)/2}\,\mathrm{i}^{-(H-Z)} \\ &L^{(+)}_{12}=\left[\frac{(q-q^{-1})(\mathrm{i}-\mathrm{i}^{-1})}{-\mathrm{i}q}\right]^{1/2} X^{+}s^{-H/2}q^{-H/2}\,\mathrm{i}^{H/2}\,\mathrm{i}^{-Z/2} \\ &L^{(-1)}_{21}=-\left[\frac{(q-q^{-1})(\mathrm{i}-\mathrm{i}^{-1})}{-\mathrm{i}q}\right]^{1/2} X^{-}s^{-H/2}q^{H/2}\,\mathrm{i}^{-H/2}\,\mathrm{i}^{Z/2}. \end{aligned} \tag{16}$$

Defining $\bar{H}\equiv H-(1+2\mathrm{i}\eta/\pi)Z$ the substitution of (16) in (13) gives $(q\equiv e^{\eta})$

$$\begin{aligned} &(X^{\pm})^2=0 \qquad [X^+, X^-]=\frac{\mathrm{i}^{\bar{H}}-\mathrm{i}^{-\bar{H}}}{\mathrm{i}-\mathrm{i}^{-1}} \\ &\Delta(X^{\pm})=\mathrm{i}^{-\bar{H}/2}s^{\pm Z/2}\otimes X^{\pm}+X^{\pm}\otimes \mathrm{i}^{\bar{H}/2}s^{\mp Z/2} \qquad S(X^{\pm})=\mp\mathrm{i}X^{\pm} \qquad S(\mathrm{i}^{\pm\bar{H}})=\mathrm{i}^{\mp\bar{H}} \end{aligned} \tag{17}$$

Note that for $s=1$ all the relations in (17) except $(X^{\pm})^2=0$ are those of $U_p(sl(2,\mathbb{C}))$ when $p=i$ (the generators being $X^{\pm}$ and $\bar{H}$). This connection is consistent with results [7] obtained previously where it was shown that $\check{R}(q, s=1)$ is associated to the two-dimensional highest weight representation of $U_p(sl(2,\mathbb{C}))$ when $p=i$. For sake of comaring with $U_{q,s}(gl(1,1))$ it will prove useful to consider a further change of basis

$$\psi^{+}=\frac{-i\sqrt{2}X^{+}i^{\bar{H}/2}}{(1-q^2)^{1/2}} \qquad \psi^{-}=\frac{i\sqrt{2}X^{-\bar{H}/2}}{(1-q^2)^{1/2}} \tag{18}$$

$\bar{U}_q(gl(1,1))$ is therefore an algebra whose generators H, ψ^+, ψ^- and Z satisfy the following relations

$$\begin{aligned} &[H,\psi^{\pm}]=\pm 2\psi^{\pm} \qquad [Z,\psi^{\pm}]=[H,Z]=0 \\ &(\psi^{\pm})^2=0 \qquad \psi^{+}\psi^{-}+\psi^{-}\psi^{+}=\frac{q^{2Z}-1}{q^2-1} \\ &\Delta(\psi^{\pm})=s^{\pm Z/2}\otimes\psi^{\pm}+\psi^{\pm}\otimes i^{(H-Z)}q^{Z}s^{\mp Z/2} \qquad S(\psi^{\pm})=-\psi^{\pm}\,i^{-(H-Z)}q^{-Z}. \end{aligned} \tag{19}$$

In verifying that (19) follows from (18) we use the fact that $i^{2\bar{H}}=(-1)^{\bar{H}}=(-1)^{H-Z}q^{2Z}$. Now $L_{11}^{(+)}L_{11}^{(-)}=s^{-(H-Z)}=(-1)^{-(H-Z)}$ for $s=-1$. It follows from the duality condition (11) that $L_{11}^{(+)}L_{11}^{(-)}$ is identical to the unit element $\mathbf{1}'$ for $s=-1$. Thus $i^{2\bar{H}}=q^{2Z}$. We stress that the coproduct is an algebra homomorphism provided one uses the rule (8) and for the antipode we have $S(AB)=S(B)S(A)$. It is important to notice that the coproduct does not become cocommutative when $q=s=1$ ($i^{(H-Z)}$ does not reduce to unity in such a limit) which exhibits the fact that this symmetry does not have a classical limit (its relation to $U_{p=i}(sl(2,\mathbb{C}))$ is also a manifestation of this). We now turn to $U_{q,s}(gl(1,1))$.

We will assign a Z_2-degree to the elements of the algebras considered. The degree of an element b will be denoted $\hat{b}$. Let $\hat{x}_1=\hat{x}^2=\hat{t}_1^1=\hat{t}_2^2=0$ and $\hat{x}^1=\hat{x}_2=\hat{t}_1^2=\hat{t}_2^1=1$. Let us now consider the case where duality with respect to the quadratic algebra B described in (4) is defined through the following pairings

$$\langle x^n; x_m\rangle=\delta_m^n \qquad \langle x^k\otimes x^l; x_m\otimes x_n\rangle=(-1)^{\hat{x}^l\hat{x}_m}\delta_m^k\delta_n^l. \tag{20}$$

Denoting by $\tilde{B}$ the dual of B it follows from (20) that

$$\tilde{B}=k\langle x^1,x^2\rangle/((x^1)^2, x^1x^2-q^{-1}s^{-1}x^2x^1). \tag{21}$$

A bialgebra with generators t_m^n ($m, n=1,2$) which coacts on the pair $(A,\tilde{B})$ can now be defined by requiring that the maps (7) be homomorphisms of A and $\tilde{B}$ respectively. Choosing the multiplication rule in the tensor product of two algebras to be

$$(a\otimes b)(c\otimes d)=(ac\otimes bd)(-1)^{\hat{b}\hat{c}} \tag{22}$$

it follows that the quadratic relations satisfied by the t_m^n's are the graded version of (9) which one obtains from (9) by making the following substitutions

$$P\to\tilde{P} \qquad R\to\tilde{R} \tag{23}$$

where

$$\tilde{P}=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&-1\end{bmatrix}$$

is the graded permutation operator and $\tilde{R} = \tilde{P}\check{R}(q, s)$ is a solution of the graded Yang–Baxter equation. We denote by $\tilde{A}_R$ the associative algebra over $\mathbb{C}$ with generators 1 and t^n_m $(m, n = 1, 2)$ which satisfy the graded version of (9). The coproduct is defined as in (10) and is an algebra homomorphisms provided one uses the rule (22). In order to define $U_{q,s}(gl(1, 1))$ we now consider the space dual to $\tilde{A}_R$. $U_{q,s}$ is defined as a subalgebra of the dual to $\tilde{A}_R$ which is generated by the unit element $1'$ and the generators $L^{(\pm)}_{kl}(k, l = 1, 2)$. The duality condition that defines $U_{q,s}$ as well as the quadratic relations satisfied by the generators are the graded versions of (11) and (13) which are obtained through the substitutions described in (23). In deriving these results one follows essentially the same arguments as the ones used by Fadeev *et al* in establishing (13) but here one exploits the fact that $\tilde{R}$ is a solution of the graded Yang–Baxter equation. See earlier works for more details [5, 8]. The coproduct and antipode are as defined in (14); however, they are algebra homomorphisms provided one uses the rule (22) and $S(AB) = S(B)S(A)(-1)^{\hat{A}\hat{B}}$. The parity is $\hat{L}^{(\pm)}_{11} = \hat{L}^{(\pm)}_{22} = 0$ and $\hat{L}^{(+)}_{12} = \hat{L}^{(-)}_{21} = 1$. The fundamental representations of $\tilde{A}_R$ and $U_{q,s}$ are the graded version of those of A_R and $\bar{U}_{q,s}(R \to \tilde{R})$.

Consider the generators H, ψ^+, ψ^- and Z such that $[H, \psi^\pm] = \pm 2\psi^\pm$ and $[Z, \psi^\pm] = [Z, H] = 0$ and make the following identification

$$\begin{aligned}
&L^{(+)}_{11} = q^{-(H+Z)/2}s^{-(H-Z)/2} && L^{(+)}_{22} = q^{-(H-Z)/2}s^{-(H+Z)/2} \\
&L^{(-)}_{11} = q^{(H+Z)/2}s^{-(H-Z)/2} && L^{(-)}_{22} = q^{(H-Z)/2}s^{-(H+Z)/2} \\
&L^{(+)}_{12} = (q - q^{-1})\psi^+ s^{-H/2} q^{-(H+Z)/2} && L^{(-)}_{21} = (q - q^{-1})\psi^- s^{-H/2} q^{(H-Z)/2}
\end{aligned} \tag{24}$$

It follows from (24) that $U_{q,s}(gl(1, 1))$ is generated by H, $\psi^\pm$ and Z which satisfy the following relations

$$\begin{gathered}
[H, \psi^\pm] = \pm 2\psi^\pm \qquad [Z, \psi^\pm] = [Z, H] = 0 \qquad (\psi^\pm)^2 = 0 \\
\psi^+\psi^- + \psi^-\psi^+ = \frac{q^{2Z} - 1}{q^2 - 1} \\
\Delta(\psi^\pm) = s^{\pm Z/2} \otimes \psi^\pm + \psi^\pm \otimes s^{\mp Z/2} q^Z \qquad S(\psi^\pm) = -\psi^\pm q^{-Z}
\end{gathered} \tag{25}$$

We stress that the coproduct is an algebra homomorphism provided one uses (22) $(\hat{\psi}^+ = \hat{\psi}^- = 1, \hat{H} = \hat{Z} = 0)$. In contrast to the previous case the coproduct is cocommutative in the limit $q = s = 1$. Note that s appears only in the coproduct, a phenomenon which has been observed in an earlier work on GL(2) [15]. The two parameter deformation of GL(1, 1) has been recently discussed by Dabrowski and Wang [16]. The graded versions of (9) and (13) are equivalent to the relations they give (see their equations (8) and (19)). They express $U_{q,s}(gl(1, 1))$ in a different basis and do not discuss $\bar{U}_{q,s}$. Equations (19) and (25) show that as algebras $U_{q,s}$ and $\bar{U}_{q,s}$ are isomorphic but differ in their coproduct and antipode (they have different Hopf structures). This is to be expected since the multiplication rules (8) and (22) are different.

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